Taylor and Maclaurin Series:

Why would we want to express a function f as a power series?

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$$

1. Lots of complicated functions can be expressed as a power series.

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}$$

2. If $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n$ then

$$f'(x) = \sum_{n=1}^{\infty} c_n \cdot n (x-a)^{n-1} \quad \text{and}$$

$$\int f(x)\,dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + k$$

3. $$\frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{\left(-\frac{x^2}{2}\right)^n}{n!}$$

$$f(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}},$$ standard normal curve.

4. Suppose $f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n =$

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + c_4(x-a)^4 + c_5(x-a)^5 + c_6(x-a)^6 + \cdots$$

Then

$$f'(x) = c_1 + 2c_2(x-a) + 3c_3(x-a)^2 + 4c_4(x-a)^3 + \cdots$$

$$f''(x) = 2c_2 + 3\cdot 2\, c_3(x-a) + 4(3)c_4(x-a)^2 + \cdots$$

$$f'''(x) = 3\cdot 2\, c_3 + 4\cdot 3\cdot 2\, c_4(x-a) + 5\cdot 4\cdot 3\, c_5(x-a)^2 + 6\cdot 5\cdot 4\, c_6(x-a)^3 + \cdots$$

Now $f(a) = c_0$

$f'(a) = c_1$

$f''(a) = 2c_2$

$f'''(a) = 3\cdot 2\, c_3$

$\vdots$

$f^{(n)}(a) = n!\, c_n$

For all n, $c_n = \dfrac{f^{(n)}(a)}{n!}$.

Definition: Taylor Series and Maclaurin Series

Given a function f with derivatives of all orders throughout an open interval containing a, the power series

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \frac{f'''(a)}{3!}(x-a)^3 + \cdots$$

is called the **Taylor series generated by f about a**. The Taylor series generated by f about 0 is also known as the **Maclaurin series generated by f**.

Definition: Taylor Polynomial of Order *n*

Given a function f with derivatives up through order N throughout an open interval containing a, the polynomial

$$p_n(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

$$= f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

for $n \in \{0, 1, \ldots, N\}$

is called the **Taylor polynomial of order n generated by f about a**. If $a = 0$, the Taylor polynomial is also known as the **Maclaurin polynomial**.

Example 1. Find the Taylor series expansion of $f(x) = \dfrac{1}{x}$ about $x = 4$ and determine where it converges to f.

$f(x) = \frac{1}{x} = x^{-1}$, $f'(x) = -x^{-2}$, $f''(x) = 2x^{-3}$,

$f^{(3)}(x) = -3 \cdot 2x^{-4}$, $f^{(4)}(x) = 4 \cdot 3 \cdot 2x^{-5}$, ...

$f(4) = \frac{1}{4}$, $f'(4) = \frac{-1}{4^2}$, $f''(4) = \frac{2!}{4^3}$,

$f^{(3)}(4) = -\frac{3!}{4^4}$, $f^{(4)}(4) = \frac{4!}{4^5}$, ...

$$f^{(n)}(4) = (-1)^n \frac{n!}{4^{n+1}}$$

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(4)}{n!}(x-4)^n = \sum_{n=0}^{\infty} \frac{(-1)^n}{4^{n+1}}(x-4)^n$$

This is a geometric series with $a = \frac{1}{4}$ and $r = -\frac{(x-4)}{4}$. It converges for $-1 < r < 1$, $-1 < -\frac{(x-4)}{4} < 1$,

$-4 < -x+4 < 4$, $0 < x < 8$.

Note that $\frac{a}{1-r} = \frac{1}{x}$.

Example 2. Find the Maclaurin polynomials for $f(x) = e^x$.

Note that $f^{(n)}(x) = e^x$, $f^{(n)}(0) = 1$

for all x.

$$\sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

Theorem: Taylor's Theorem

If f and its derivatives up through $f^{(n)}$ are all continuous on the closed interval $[a, b]$ and if $f^{(n+1)}$ exists on the open interval (a, b), then there is a number $c \in (a, b)$ such that

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2!}(b-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(b-a)^n + \frac{f^{(n+1)}(c)}{(n+1)!}(b-a)^{n+1}.$$

Formula: Taylor's Formula

If f has derivatives of all orders throughout an open interval I containing a, then for each natural number n and for each $x \in I$,

$$f(x) = \underbrace{f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n}_{p_n(x)} + \underbrace{\frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}}_{r_n(x)}$$

where c is some number between a and x. Defining $p_n(x)$ and $r_n(x)$ as above, we can write $f(x) = p_n(x) + r_n(x)$, where p_n is a polynomial of degree at most n and r_n is called the **remainder** (or **error) of order n** .

Example 3. Show that for all x, $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$.

$$r_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!} x^{n+1} = e^c \cdot \frac{x^{n+1}}{(n+1)!}$$

Recall that the ratio test show that $\sum_{n=0}^{\infty} \frac{x^n}{n!}$ converges for all x

and so $\lim_{n \to \infty} \frac{x^n}{n!} = 0$.

Now, $\lim_{n \to \infty} r_n(x) = \lim_{n \to \infty} e^c \frac{x^{n+1}}{(n+1)!} = 0$.

Therefore $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ for all x.

Example 4. Show that for all x, $\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^n}{n!}$.

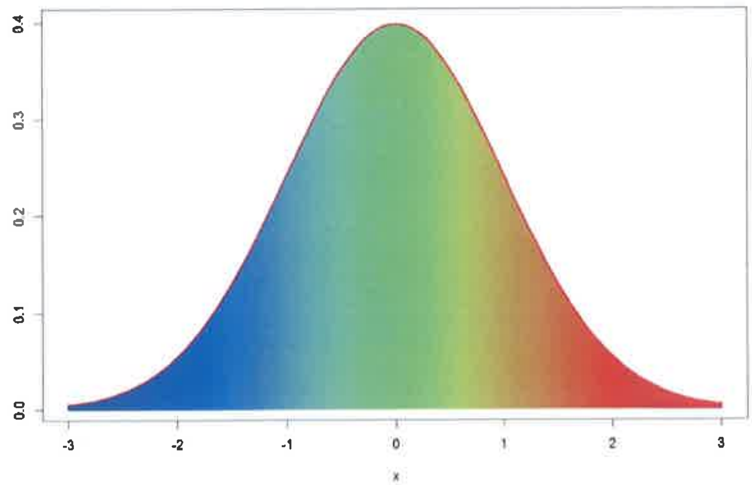
By Taylor's theorem,

$r_n(x) = \frac{f^{(n+1)}(c)\, x^{n+1}}{(n+1)!}$. The derivatives

of $\sin x$ are all $\pm \sin x$, $\pm \cos x$ &

so $|r_n(x)| \leq \frac{|x|^{n+1}}{(n+1)!} \longrightarrow 0$ as $n \to \infty$.

Example 5. The famous normal curve from probability and statistics is given by $f(x)=\frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. The chance that an event will occur is often given by $\int_a^b \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}\,dx$. How are such integrals evaluated?



$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}, \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} = \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \left(-\frac{x^2}{2}\right)^n / n!$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^n \cdot n!}$$

$$\int_a^b \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx \approx \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{k} \left[\frac{(-1)^n x^{2n+1}}{2^n \cdot n! \, (2n+1)}\right]\Bigg|_a^b$$

Example 6. Find the Taylor series approximation of the definite integral $\int_0^1 \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\,dx$ that guarantees an error of at most 0.01.

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n \, 1^{2n+1}}{2^n \cdot n! \, (2n+1)}$$

$$= \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{4} (-1)^n \frac{1}{2^n \cdot n! \, (2n+1)}$$

when $n=4$, $|a_5| = .000023$ so use

$$\frac{1}{\sqrt{2\pi}} \left[1 - \frac{1}{6} + \frac{1}{8(5)} - \frac{1}{48(7)} + \frac{1}{16(24)(9)} \right]$$

$$= \frac{1}{\sqrt{2\pi}} (.855649)$$

$$\approx \boxed{.34135}$$

Accurate to 3 decimal places!

Example 7. Find the Taylor series approximation of $\ln(1.17)$ that guarantees an error of at most 10^{-3}.

We need the Taylor series expansion of $\ln x$ about $x=1$.

$$\frac{1}{x} = \frac{1}{1-(1-x)} = \frac{1}{1-r} = \sum_{n=0}^{\infty} r^n \quad \text{where}$$

$r = 1-x$ and $|r| = |1-x| < 1$, $0 < x < 2$

$$\text{so } \frac{1}{x} = \sum_{n=0}^{\infty} (1-x)^n,$$

$$\ln x = \sum_{n=0}^{\infty} -\frac{(1-x)^{n+1}}{n+1} = -\sum_{n=0}^{\infty} (-1)^{n+1} \frac{(x-1)^{n+1}}{n+1}$$

$$= \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(x-1)^n}{n}.$$

$$\ln(1.17) = \sum_{n=1}^{\infty} (-1)^{n+1} \frac{(.17)^n}{n}$$

We want $|a_{n+1}| = \dfrac{(.17)^{n+1}}{n+1} < \dfrac{1}{10^4}$.

Choose $n=4$.

$$\ln(1.17) \approx (.17) - \frac{(.17)^2}{2} + \frac{(.17)^3}{3} - \frac{(.17)^4}{4}$$

$$\approx .156978$$

$$\approx .157 \quad \text{(3 decimal places)}$$

$$\ln(1.17) = .157004$$

(6 decimals)